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Generalized Complex Spherical Harmonics, Frame Functions, and Gleason Theorem

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Abstract. Consider a finite dimensional complex Hilbert space $\mathcal{H}$, with $dim(\mathcal{H}) \geq 3$, define $\mathbb{S}(\mathcal{H}) := \{x \in \mathcal{H} \mid ||x|| = 1\}$, and let $\nu_{\mathcal{H}}$ be the unique regular Borel positive measure invariant under the action of the unitary operators in $\mathcal{H}$, with $\nu_{\mathcal{H}}(\mathbb{S}(\mathcal{H})) = 1$. We prove that if a complex frame function $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$ satisfies $f \in \mathcal{L}^2(\mathbb{S}(\mathcal{H}), \nu_{\mathcal{H}})$, then it verifies Gleason's statement: There is a unique linear operator $A : \mathcal{H} \to \mathcal{H}$ such that $f(u) = \langle u|Au\rangle$ for every $u \in \mathbb{S}(\mathcal{H})$. A is Hermitean when f is real. No boundedness requirement is thus assumed on f *a priori*.

1 Introduction

In the absence of superselection rules, the states of a quantum system described in the Hilbert space $\mathcal{H}$ are defined as generalized probability measures $\mu : \mathfrak{P}(\mathcal{H}) \to [0, 1]$ on the lattice $\mathfrak{P}(\mathcal{H})$ of orthogonal projectors in $\mathcal{H}$. By definition μ is required to verify (i) $\mu(I) = 1$ and (ii) $\mu(\sum_{k\in K} P_k) = \sum_{k\in K} \mu(P_k)$, where $\{P_k\}_{k\in K} \subset \mathfrak{P}(\mathcal{H})$, with K finite or countable, is any set satisfying $P_iP_j = 0$ for $i \neq j$ and the sum in the right-hand side in (ii) is computed respect to the strong operator topology if K is infinite.

Normalized, positive trace-class operators, namely *density* or *statistical* operators, very familiar to physicists, define such measures. However, the complete characterization of those measures was established by Gleason [Gle57], with a milestone theorem whose proof is unexpectedly difficult.

***Theorem* 1** **(Gleason's theorem)** *Let $\mathcal{H}$ be a (real or complex) separable Hilbert space with $3 \leq dim(\mathcal{H}) \leq +\infty$. For every generalized probability measures $\mu : \mathfrak{P}(\mathcal{H}) \to [0, 1]$, there exist a unique positive, self-adjoint trace class operator T_μ, with unit trace, such that:*

$$\mu(P) = tr(T_\mu P) \quad \forall P \in \mathfrak{P}(\mathcal{H}).$$

The key-tool exploited in Gleason's proof is the notion of *frame function* that will be the object of this paper.

Definition 2 *Let $\mathcal{H}$ be a complex Hilbert space and $\mathbb{S}(\mathcal{H}) := \{\psi \in \mathcal{H} \mid ||\psi|| = 1\}$. $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$ is a* **frame function** *on $\mathcal{H}$ if $W_f \in \mathbb{C}$ exists, called* ***weight*** *of f, with:*

$$\sum_{x \in N} f(x) = W_f \quad \textit{for every Hilbertian basis } N \textit{ of } \mathcal{H}. \tag{1}$$

(If $\mathcal{H}$ is non-separable, the series is the integral with respect to the measure counting the points of N.)

With the hypotheses of Gleason's theorem, the restriction f_μ of μ to the set of the projectors on one-dimensional subspaces is a *real* and *bounded* frame function. It is known that on a real Hilbert space $\mathcal{H}$ with $dim(\mathcal{H}) = 3$, a frame function which is bounded (even from below only or only from above only) is continuous and can be uniquely represented as a quadratic form [Gle57, Dvu92]. That result is very difficult to be established and is the kernel of the original proof of the Gleason theorem. The last non-trivial step in Gleason's proof is passing from 3 real dimensions to any (generally complex) dimension, this is done exploiting Riesz theorem, establishing that there is a unique positive, self-adjoint trace-class operator T_μ with $tr(T_\mu) = 1$ such that $f_\mu(x) = \langle x|T_\mu x\rangle$ for all $x \in \mathbb{S}(\mathcal{H})$. The final step is the easiest one: if $P \in \mathfrak{P}(\mathcal{H})$, there is a Hilbert basis N such that in the strong operator topology $P = \sum_{z \in N_P} z\langle z|\cdot\rangle$ for some $N_P \subset N$. So that $\mu(P) = \sum_{z \in N_P} f(z) = tr(PT_\mu)$.

Frame functions are therefore remarkable tools to manipulate generalized measures. However, they are interesting on their own right [Dvu92]. An important difference, distinguishing the finite-dimensional case from the infinite-dimensional one, is that a frame function on an *infinite* dimensional Hilbert space has to be automatically bounded [Dvu92]. Whereas in the finite-dimensional case ($dim(\mathcal{H}) \geq 3$), as proved by Gudder and Sherstnev, there exist infinitely many unbounded frame functions [Dvu92]. The bounded ones are the only representable as quadratic forms.

In the rest of the paper we prove a proposition concerning sufficient conditions to assure that a frame function, on a complex finite-dimensional Hilbert space $\mathcal{H}$, with $dim(\mathcal{H}) \geq 3$, is representable as a quadratic form without assuming the boundedness requirement *a priori*. Instead we treat the topic from another point of view. The sphere $\mathbb{S}(\mathcal{H})$, up to a multiplicative constant, admits a unique regular Borel measure invariant under the action of all unitary operators in $\mathcal{H}$. We prove that, for $dim(\mathcal{H}) \geq 3$, a complex frame function f is representable as a quadratic form whenever it is Borel-measurable and belongs to $\mathcal{L}^2(\mathbb{S}(\mathcal{H}), \nu_\mathcal{H})$. In particular it holds when $f \in \mathcal{L}^p(\mathbb{S}(\mathcal{H}), \nu_\mathcal{H})$ for some $p \in [2, +\infty]$. The proof is direct and relies upon the properties of the spaces of generalized complex spherical harmonics [Rud86] and on some results due to Watanabe [Wat00] on zonal harmonics, beyond standard facts on Hausdorff compact topological group representations (the classic Peter-Weyl theorem).

2 Generalized Complex Spherical Harmonics

Let us introduce a n-dimensional generalization of spherical harmonics defined on:

$$\mathbb{S}^{2n-1} := \{x \in \mathbb{C}^n \mid ||x|| = 1\} . \tag{2}$$

$\mathbb{S}^{2n-1} \subset \mathbb{R}^{2n}$ is, in fact, a $2n-1$-dimensional real smooth manifold.
ν_n denotes the $U(n)$-left-invariant regular Borel measure on $\mathbb{S}^{2n-1}$, normalized to $\nu_n(\mathbb{S}^{2n-1}) = 1$, obtained from the two-sided Haar measure on $U(n)$ on the homogeneous space given by the quotient $U(n)/U(n-1) \equiv \mathbb{S}^{2n-1}$. That measure exists and is unique as follows from general results by Mackey (e.g., see Chapter 4 of [BR00], noticing that both $U(n)$ and $U(n-1)$ are compact and thus unimodular).

Lemma 3 *$\nu_n(A) > 0$ if $A \neq \emptyset$ is an open subset of $\mathbb{S}^{2n-1}$.*

Proof. $\{gA\}_{g\in U(n)}$ is an open covering of $\mathbb{S}^{2n-1}$. Compactness implies that $\mathbb{S}^{2n-1} = \cup_{k=1}^{N} g_k A$ for some finite N. If $\nu_n(A) = 0$, sub-additivity and $U(n)$-left-invariance would imply $\nu_n(\mathbb{S}^{2n-1}) = 0$ that is false. $\square$

As ν_n is $U(n)$-left-invariant,

$$U(n) \ni g \to D_n(g) \quad \text{with } D_n(g)f := f \circ g^{-1} \text{ for } f \in L^2(\mathbb{S}^{2n-1}, d\nu_n) \tag{3}$$

defines a faithful unitary representation of $U(n)$ on $L^2(\mathbb{S}^{2n-1}, d\nu_n)$.

Lemma 4 *For every $n = 1, 2, \ldots$ the unitary representation (3) is strongly continuous.*

Proof. It is enough proving the continuity at $g = I$. If $f : \mathbb{S}^{2n-1} \to \mathbb{C}$ is continuous, $U(n) \times \mathbb{S}^{2n-1} \ni (g, u) \mapsto f(g^{-1}u)$ is jointly continuous and thus bounded by $K < +\infty$ since the domain is compact. Exploiting Lebesgue dominated convergence theorem as $|f \circ g^{-1}(u) - f(u)|^2 \leq K$ and the constant function K being integrable since the measure ν_n is finite:

$$||D_n(g)f - f||_2^2 = \int_{\mathbb{S}^{2n-1}} |f \circ g^{-1} - f|^2 d\nu_n \to 0 \quad \text{as } g \to I,$$

If f is not continuous, due to Luzin's theorem, there is a sequence of continuous functions f_n converging to f in the norm of $L^2(\mathbb{S}^{2n-1}, d\nu_n)$. Therefore

$$||f \circ g^{-1} - f||_2 \leq ||f \circ g^{-1} - f_n \circ g^{-1}||_2 + ||f_n \circ g^{-1} - f_n||_2 + ||f_n - f||_2 .$$

If $\epsilon > 0$, there exists k with $||f \circ g^{-1} - f_k \circ g^{-1}||_2 = ||f - f_k||_2 < \epsilon/3$ where we have also used the $U(n)$-invariance of ν_n. Since f_k is continuous we can apply the previous result getting $||f_k \circ g^{-1} - f_k||_2 < \epsilon/3$ if g is sufficiently close to I. $\square$

We are in a position to define the notion of spherical harmonics we shall use in the rest of the paper. If, $p, q = 0, 1, 2, \ldots$, $\mathcal{P}^{p,q}$ denotes the set of polynomials $h : \mathbb{S}^{2n-1} \to \mathbb{C}$ such that $h(\alpha z_1, ..., \alpha z_n) = \alpha^p \overline{\alpha}^q h(z_1, ..., z_n)$ for all $\alpha \in \mathbb{C}$. The standard Laplacian Δ_{2n} on $\mathbb{R}^{2n}$ can be applied to the elements of $\mathcal{P}^{p,q}$ in terms of decomplexified $\mathbb{C}^n$. Now, we have the following known result (see Theorems 12.2.3, 12.2.7 in [Rud86] and theorem 1.3 in [JW77]):

***Theorem* 5** *If $\mathcal{H}^n_{(p,q)} := Ker\Delta_{2n}|_{\mathcal{P}^{p,q}}$, the following facts hold.*
(a) *The orthogonal decomposition is valid, each $\mathcal{H}^n_{(p,q)}$ being finite-dimensional and closed:*

$$L^2(\mathbb{S}^{2n-1}, d\nu_n) = \bigoplus_{p,q=0}^{+\infty} \mathcal{H}^n_{(p,q)}. \tag{4}$$

(b) *Every $\mathcal{H}^n_{(p,q)}$ is invariant and irreducible under the representation (3) of $U(n)$, so that the said representation correspondingly decomposes as*

$$D_n(g) = \bigoplus_{p,q=0}^{+\infty} D_n^{(p,q)}(g) \quad \text{with } D_n^{(p,q)}(g) := D_n(g)|_{\mathcal{H}^n_{(p,q)}}.$$

(c) *If $(p, q) \neq (r, s)$ the irreducible representations $D_n^{p,q}$ and $D_n^{r,s}$ are unitarily inequivalent: no unitary operator $U : \mathcal{H}^n_{(p,q)} \to \mathcal{H}^n_{(r,s)}$ exists such that $UD_n^{(p,q)}(g) = D_n^{(r,s)}(g)U$ for every $g \in U(n)$.*

***Definition* 6** *For $j \equiv (p, q)$, with $p, q = 0, 1, 2, ...$, the* **generalized complex spherical harmonics** *of order j are the elements of $\mathcal{H}^n_{(p,q)}$.*

A useful technical lemma is the following.

***Lemma* 7** *For $n \geq 3$, $\mathcal{H}^n_{(1,1)}$ is made of the restrictions to $\mathbb{S}^{2n-1}$ of the polynomials $h^{(1,1)}(z, \overline{z}) = \overline{z}^t Az$, A being any traceless $n \times n$ matrix and $z \in \mathbb{C}^n$.*

Proof. $h^{(1,1)}$ is of first-degree in each variables so $h^{(1,1)}(z, \overline{z}) = \overline{z}^t Az$ for some $n \times n$ matrix A. $\Delta_{2n} h^{(1,1)} = 0$ is equivalent to $trA = 0$ as one verifies by direct inspection. □

For $n \geq 3$, there is a special class of spherical harmonics in $\mathcal{H}^n_j$ that are parametrized by elements $t \in \mathbb{S}^{2n-1}$ [Wat00].

Definition* 8** *For $n \geq 3$, the* ***zonal spherical harmonics *are elements of $\mathcal{H}^n_j$ defined, for every $t \in \mathbb{C}^n$, as*

$$F^j_{n,t}(u) := R^n_j \left(\overline{u}^t \cdot t\right) \quad \forall u \in \mathbb{S}^{2n-1}, \tag{5}$$

where the polynomials $R^n_j(z)$ have the generating function

$$(1 - \xi z - \eta \overline{z} + \xi\eta)^{1-n} = \sum_{p,q=0}^{+\infty} R^n_{p,q}(z) \xi^p \eta^q \tag{6}$$

with $|z| \leq 1$, $|\eta| < 1$, $|\xi| < 1$.

These zonal spherical harmonics are a generalization of the eigenfunctions of orbital angular momentum with L_z-eigenvalue $m = 0$. From (6) we get two identities useful later:

$$p!q!R^n_{p,q}(1) = (-1)^{p+q}(n-1)n(n+1)\cdots(n+p-2)(n-1)n(n+1)\cdots(n+q-2),$$

$$p!q!R^n_{p,q}(0) = (-1)^p\delta_{pq}p!(n-1)n(n+1)\cdots(n+p-2). \tag{7}$$

3 Generalized complex Harmonics and Frame Functions

To prove our main statement in the next section we need the following preliminary technical result that relies on the technology presented in Chapter 7 of [BR00].

***Proposition* 9** *If $f \in \mathcal{L}^2(\mathbb{S}^{2n-1}, d\nu_n)$, each projection f_j on $\mathcal{H}^n_j$ verifies, μ being the Haar measure on $U(n)$ normalized to $\mu(U(n)) = 1$:*

$$f_j(u) = dim(\mathcal{H}^n_j)\int_{U(n)} tr(\overline{D^j(g)})f(g^{-1}u)d\mu(g) \quad \textit{a.e. in } u \textit{ with respect to } \nu_n, \tag{8}$$

where the right-hand side is a continuous function of $u \in \mathbb{S}^{2n-1}$.
If $f \in \mathcal{L}^2(\mathbb{S}^{2n-1}, d\nu_n)$ is a frame function, then f_j (possibly re-defined on a zero-measure set in order to be continuous) is a frame function as well with $W_{f_j} = 0$ when $j \neq (0,0)$.

Proof. First of all notice that, if $f \in \mathcal{L}^2(\mathbb{S}^{2n-1}, d\nu_n)$, the right-hand side of (8) is well defined and continuous as we go to prove. $U(n) \ni g \mapsto tr(\overline{D^j(g)})$ is continuous – and thus bounded since $U(n)$ is compact – in view of lemma 4 and $dim(\mathcal{H}^n_j)$ is finite for theorem 5. Furthermore, for almost all $u \in \mathbb{S}^{2n-1}$ the map $U(n) \ni g \mapsto f(g^{-1}u)$ is $\mathcal{L}^2(U(n), d\mu)$ – and thus $\mathcal{L}^1(U(n), d\mu)$ because the measure is finite – as follows by Fubini-Tonelli theorem and the invariance of ν_n under $U(n)$, it being

$$\int_{U(n)} d\mu(g)\int_{\mathbb{S}^{2n-1}} |f(g^{-1}u)|^2 d\nu_n(u) = \int_{U(n)} d\mu(g)\int_{\mathbb{S}^{2n-1}} |f(u)|^2 d\nu_n(u) = \mu(U(n))||f||_2 < +\infty.$$

Consequently, in view of the fact that μ is invariant, and $U(n)$ transitively acts on $\mathbb{S}^{2n-1}$, the map $U(n) \ni g \mapsto f(g^{-1}u)$ is $\mathcal{L}^2(U(n), d\mu)$ (and thus $\mathcal{L}^1(U(n), d\mu)$) for all $u \in \mathbb{S}^{2n-1}$. Continuity in u of the right-hand side of (8) can be proved as follows. Let $u_0 = [I] \in \mathbb{S}^{2n-1} \equiv U(n)/U(n-1)$. Since $U(n)$ and $U(n-1)$ are Lie groups, for any fixed $u_1 \in \mathbb{S}^{2n-1}$ there is an open neighbourhood W_{u_1} of u_1 and a smooth map $W_{u_1} \ni u \mapsto g_u \in U(n)$ such that $[g_u] = u$ (Theorem 3.58 in [War83]). As a consequence $g_u u_0 = [g_u I] = [g_u] = u$. Therefore, using the invariance of the Haar measure and for $u = g_u u_0 \in W_{u_1}$:

$$\int_{U(n)} tr(\overline{D^j(g)})f(g^{-1}u)d\mu(g) = \int_{U(n)} tr(\overline{D^j(g_u g)})f(g^{-1}u_0)d\mu(g)\,.$$

Since $W_{u_1} \times U(n) \ni (u, g) \mapsto tr(\overline{D^j(g_u g)})$ is continuous due to lemma 4, the measure is finite and $g \mapsto f(g^{-1}u_0)$ is integrable, Lebesgue dominated convergence theorem implies that, as said above, $W_{u_1} \ni u \mapsto \int_{U(n)} tr(\overline{D^j(g_u g)}) f(g^{-1}u_0) d\mu(g)$ is continuous in u_1 and thus everywhere on $\mathbb{S}^{2n-1}$ since u_1 is arbitrary.
Let us pass to prove (8) for f containing a finite number of components. So F is finite, $f \in \mathcal{L}^2(\mathbb{S}^{2n-1}, d\nu_n)$ and:

$$f(u) = \sum_{j \in F} f_j(u) = \sum_{j \in F} \sum_{m=1}^{dim(\mathcal{H}_j^n)} f_m^j Z_m^j(u) \,, \qquad f_m^j \in \mathbb{C}$$

where $\{Z_m^j\}_{m=1,\ldots,dim(\mathcal{H}_j^n)}$ is an orthonormal basis of $\mathcal{H}_j^n$, with $Z_n^{(0,0)} = 1$, made of continuous functions (it exists in view of the fact that $\mathcal{P}^{p,q}$ is a space of polynomials and exploiting Gramm-Schmitd's procedure). Then

$$\overline{D^{j_0}_{m_0 m_0'}(g)} f(g^{-1}u) = \sum_{j \in F} \sum_{m,m'} \overline{D^{j_0}_{m_0 m_0'}(g)} D^j_{mm'}(g) f^j_{m'} Z^j_m(u) \,.$$

In view of (c) of theorem 5 and Peter-Weyl theorem, taking the integral over g with respect to the Haar measure on $U(n)$ one has:

$$\int \overline{D^{j_0}_{m_0 m_0'}(g)} f(g^{-1}u) d\mu(g) = dim(\mathcal{H}^n_{j_0})\, f^{j_0}_{m_0'} Z^{j_0}_{m_0}(u) \,,$$

that implies (8) when taking the trace, that is summing over $m_0 = m_0'$. To finish with the first part, let us generalize the obtained formula to the case of F infinite. In the following $P_j : L^2(\mathbb{S}^{2n-1}, \nu_n) \to L^2(\mathbb{S}^{2n-1}, \nu_n)$ is the orthogonal projector onto $\mathcal{H}^n_j$. The convergence in the norm $|| \; ||_2$ implies that in the norm $|| \; ||_1$, since $\nu_n(\mathbb{S}^{2n-1}) < +\infty$. So if $h_m \to f$ in the norm $|| \; ||_2$, as P_j is bounded:

$$\lim_{m \to +\infty}{}^{(1)} P_j h_m = \lim_{m \to +\infty}{}^{(2)} P_j h_m = P_j \left(\lim_{m \to +\infty}{}^{(2)} h_m \right) = P_j f \,.$$

We specialize to the case $h_m = \sum_{(p,q)=(0,0)}^{p+q=m} f_{(p,q)}$ so that $h_m \to f$ as $m \to +\infty$ in the norm $|| \; ||_2$. As every h_m has a finite number of harmonic components the identity above yields:

$$dim(\mathcal{H}^n_j) \lim_{m \to +\infty}{}^{(1)} \int_{U(n)} tr(\overline{D^j(g)}) h_m(g^{-1}u) d\mu(g) = P_j f =: f_j \,.$$

Now notice that, as $U(n) \ni g \mapsto tr(\overline{D^j(g)})$ is bounded on $U(n)$ by some $K < +\infty$:

$$\left|\left| \int_{U(n)} tr(\overline{D^j(g)}) h_m(g^{-1}u) d\mu(g) - \int_{U(n)} tr(\overline{D^j(g)}) f(g^{-1}u) d\mu(g) \right|\right|_1$$

$$\leq K \int_{\mathbb{S}^{2n-1}} d\nu(u) \int_{U(n)} d\mu(g) \left|h_m(g^{-1}u) - f(g^{-1}u)\right|$$

$$= K \int_{U(n)} d\mu(g) \int_{\mathbb{S}^{2n-1}} d\nu(u) \left|h_m(g^{-1}u) - f(g^{-1}u)\right|$$

$$= K \int_{U(n)} d\mu(g) \int_{\mathbb{S}^{2n-1}} d\nu(u) \left|h_m(u) - f(u)\right| = K\mu(U(n))||h_m - f||_1 \to 0\,.$$

We have found that, as desired, (8) holds for f, because

$$\left|\left|f_j - dim(\mathcal{H}^n_j) \int_{U(n)} tr(\overline{D^j(g)}) f(g^{-1}u) d\mu(g)\right|\right|_1 = 0\,.$$

To prove the last statement, assume $j \neq (0,0)$ otherwise the thesis is trivial (since $f_{(0,0)}$ is a constant function). We notice that, when f_j is taken to be continuous (and it can be done in a unique way in view of lemma 3, referring the the Hilbert basis of continuous functions Z^j_m as before), the identity (8) must be true for every $u \in \mathbb{S}^{2n}$. Therfore, if $e_1, e_2, \ldots, e_n$ is any Hilbert basis of $\mathbb{C}^n$

$$\frac{1}{dim(\mathcal{H}^n_j)} \sum_k f_j(e_k) = \int_{U(n)} tr(\overline{D^j(g)}) \sum_k f(g^{-1}e_k) d\mu(g) = \int_{U(n)} tr(\overline{D^j(g)}) W_f d\mu(g) = 0$$

because W_f is a constant and thus it is proportional to $1 = D^{(0,0)}$ which, in turn, is orthogonal to $D^j_{mm'}$ for $j \neq (0,0)$ in view of Peter-Weyl theorem and (c) of theorem 5. $\Box$

4 The main result

If $\mathcal{H}$ is a finite-dimensional complex Hilbert space $\mathcal{H}$, with $\dim(\mathcal{H}) = n \geq 3$, there is only a regular Borel measure, $\nu_{\mathcal{H}}$, on $\mathbb{S}(\mathcal{H})$ which is left-invariant under the natural action of every unitary operator $U : \mathcal{H} \to \mathcal{H}$ and $\nu_{\mathcal{H}}(\mathbb{S}(\mathcal{H})) = 1$. It is the $U(n)$-invariant measure ν_n induced by any identification of $\mathcal{H}$ with a corresponding $\mathbb{C}^n$ obtained by fixing a orthonormal basis in $\mathcal{H}$. The uniqueness of $\nu_{\mathcal{H}}$ is due to the fact that different orthonormal bases are connected by means of transformations in $U(n)$.

***Theorem* 10** *If $f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$ is a frame function on a finite-dimensional complex Hilbert space $\mathcal{H}$, with $\dim(\mathcal{H}) \geq 3$ and $f \in \mathcal{L}^2(\mathbb{S}(\mathcal{H}), d\nu_{\mathcal{H}})$, then there is a unique linear operator $A : \mathcal{H} \to \mathcal{H}$ such that:*

$$f(z) = \langle z|Az\rangle \quad \forall z \in \mathbb{S}(\mathcal{H}), \tag{9}$$

where $\langle\ |\ \rangle$ is the inner product in $\mathcal{H}$. A turns out to be Hermitean if f is real.

Remark. Since $\nu_{\mathcal{H}}$ is finite, $f \in \mathcal{L}^2(\mathbb{S}(\mathcal{H}), d\nu_{\mathcal{H}})$ holds in particular when $f \in \mathcal{L}^p(\mathbb{S}(\mathcal{H}), d\nu_{\mathcal{H}})$ for some $p \in [2, +\infty]$, as a classic result based on Jensen's inequality.

Proof. We start from the uniqueness issue. Let B be another operator satisfying the thesis, so that $\langle z|(A-B)z\rangle = 0\ \forall z \in \mathcal{H}$. Choosing $z = x+y$ and then $z = x+iy$ one finds $\langle x|(A-B)y\rangle = 0$ for every $x, y \in \mathcal{H}$, that is $A = B$. We pass to the existence of A identifying $\mathcal{H}$ to $\mathbb{C}^n$ by means of an orthonormal basis $\{e_k\}_{k=1,\ldots,n} \subset \mathcal{H}$. As $f \in \mathcal{L}^2(\mathbb{S}^{2n-1}, d\nu_n)$, f can be decomposed as: $f = \sum_j f_j$ with $f_j \in \mathcal{H}^n_j$. Lemma 9 implies that, if $g \in U(n)$:

$$\sum_{k=1}^{n} \left(D^j(g) f_j\right)(e_k) = \sum_{k=1}^{n} f_j(g^{-1}e_k) = 0 \quad \text{if} \quad j \neq (0,0) \tag{10}$$

Assuming $f_j \neq 0$, since the representation D^j is irreducible, the subspace of $\mathcal{H}^n_j$ spanned by all the vectors $D^j(g)f_j \in \mathcal{H}^n_j$ is dense in $\mathcal{H}^n_j$ when g ranges in $U(n)$. As $\mathcal{H}^n_j$ is finite-dimensional, the dense subspace is $\mathcal{H}^n_j$ itself. So it must be $\sum_{k=1}^n Z(e_k) = 0$ for every $Z \in \mathcal{H}^n_j$. In particular it holds for the zonal spherical harmonic F^j_{n,e_1} individuated by e_1: $\sum_{k=1}^n F^j_{n,e_1}(e_k) = 0$. By definition of zonal spherical harmonics the above expression can be written in these terms: $R^n_{p,q}(1) + (n-1)R^n_{p,q}(0) = 0$, and using relations (7):

$$\begin{aligned}(-1)^{p+q}(n-1)n(n+1)\cdots(n+p-2)(n-1)n(n+1)\cdots(n+q-2) = \\ = (-1)^p \delta_{pq} p!(n-1)^2 n(n+1)\cdots(n+p-2).\end{aligned} \tag{11}$$

(11) implies $p = q$. Indeed, if $p \neq q$ the right hand side vanishes, while the left does not. Now, for $n \geq 3$ and $j \neq (0,0)$ we can write:

$$(n-1)^2 n^2 (n+1)^2 \cdots (n+p-2)^2 = (-1)^p p!(n-1)^2 n(n+1)\cdots(n+p-2). \tag{12}$$

The identity (12) is verified if and only if $p = 1$. In view of lemma 7, we know that the functions $f_{(1,1)} \in \mathcal{H}^n_{(1,1)}$ have form $f(x) = \langle x|A_0 x\rangle$ with $trA_0 = 0$. We conclude that our frame function f can only have the form:

$$f(x) = c + f_{(1,1)}(x) = \langle x|cIx\rangle + \langle x|A_0 x\rangle = \langle x|Ax\rangle \quad x \in \mathbb{S}^{2n-1}\,.$$

If f is real valued $\langle x|Ax\rangle = \overline{\langle x|Ax\rangle} = \langle x|A^* x\rangle$ and thus $\langle x|(A-A^*)x\rangle = 0$. Exploiting the same argument as that used in the proof of the uniqueness, we conclude that $A = A^*$. $\square$

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